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LETTER TO THE EDITOR

A characteristic of the Lyapunov spectrum for the multichannel Anderson localisation in the thermodynamical limit

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Abstract. The distribution of Lyapunov exponents of the N-channel Anderson problem is investigated. The Kolmogorov entropy and the maximum Lyapunov exponent are calculated in the thermodynamical limit at weak disorder. The ratio I of these two quantities, which characterises the convexity of the distribution, is shown to depend on the disorder, the energy and the energy spectrum of the noiseless system. Moreover, I is found to be in general greater than two, in agreement with the convex spectrum obtained previously by some authors through numerical experiments. It is also shown that, in contrast to the one-dimensional case, the 'generalised Lyapunov exponent' of order two and the largest Lyapunov exponent are equal in the limit $N \to \infty$.

Most of the analytical results concerning localisation theory have been obtained for one-dimensional systems [1]. In this case, the system is characterised by only two Lyapunov exponents, which are equal and of opposite sign. Due to the existence of a single positive Lyapunov exponent, localisation in one dimension occurs for arbitrary disorder.

In two or three dimensions, where an entire spectrum of Lyapunov exponents exists, most reported work consists of numerical simulations. Among the questions of interest, the existence of a localisation transition has been thoroughly studied leading to the belief that states are always localised in two dimensions while in three dimensions a transition occurs [2, 3, 4]. Another problem of great interest concerns the thermodynamical limit of disordered strips or bars when their transversal sizes go to infinity. More precisely Newman [5] motivated by a work of Ruelle [6] has proposed that such systems should exhibit a limiting density of Lyapunov exponents. He suggested that this limiting density may be in some sense universal and he explicitly derived this distribution in some particular cases, exhibiting the triangular law that is a regular distribution of Lyapunov exponents. Universal properties have been also considered by Imry [7] as a support for using random matrix theory in order to explain universal fluctuations of conductance [8]. On the other hand, numerical simulations have been performed by Pichard and André [9] and by Livi *et al* [10]. Both of them obtain in the thermodynamical limit a regular distribution of Lyapunov exponents. However, their results show some deviation from the triangular law especially for the largest exponents. Moreover, in the case of bars they have noted a change in the concavity of the distribution at about the level of disorder corresponding to the localisation transition in three dimensions. It is interesting to note that according to their works such a change of concavity also occurs for strips.

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With the exception of results obtained on the transport properties of samples by means of field theoretical methods, few analytical results have been otherwise obtained for two-dimensional systems and even for quasi-one-dimensional systems. In a recent paper Derrida *et al* [11] have calculated the whole spectrum of Lyapunov exponents of a product of random matrices. Unfortunately their method and results are not valid in the physical domain of energy.

It is the aim of this letter to investigate the distribution of Lyapunov exponents at low disorder in the thermodynamical limit. This letter is organised as follows. Firstly we consider a general ergodic linear and Hamiltonian dynamical system whose Hamiltonian H is: $H = H_0 + V$. V is a Gaussian white noise potential. We show that one can compute systematically a perturbative expansion for the sum of its positive Lyapunov exponents, i.e. its Kolmogorov entropy. Secondly we show that the highest positive Lyapunov exponent of the multichannel case is easily computable in the thermodynamical limit. With the help of these two results we investigate the limiting distribution of the Lyapunov spectrum in the two- and three-dimensional Anderson localisation. In particular we argue that it depends on the spectral properties of the unperturbed system.

Let us consider a linear Hamiltonian system with N degrees of freedom. Q_i and P_i, $i = 1, \ldots, N$, denote respectively the position variables and the momenta. To these variables we associated a vector: $\Phi = (Q_i, P_i)$. We write the Hamiltonian as

$$H(P, Q) = H_0(P, Q) + V(Q, t) \tag{1}$$

where H and H_0 are quadratic forms. H_0 is the unperturbed time-independent Hamiltonian. It describes a set of harmonic oscillators and is thus strictly positive (the case H_0 negative corresponds to the problem treated by Derrida *et al* [11]). V is a time-dependent multiplicative noise of zero mean. In the following, H_0 and V will also denote their associated $N \times N$ matrices. With these notations we have

$$\langle V_{ij}(t)\rangle = 0 \qquad \overline{V_{ij}(t) V_{kl}(t')} = \langle V_{ij}(t) V_{kl}(t')\rangle = D_{ijkl}\delta(t-t') \tag{2}$$

where D is a tensor of order four.

The N-channel localisation problem, investigated later in this letter, is defined by [12]:

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi_i(x) + \sum_j V_{ij}(x)\Psi_j(x) - \gamma(\Psi_{i+1}(x) - 2\Psi_i(x) + \Psi_{i-1}(x)) = E\Psi_i(x) \tag{3}$$

where x is the position along the channels, Ψ_i is the wavefunction on the ith channel and V satisfies (2) with $D_{ijkl} = D$ if $i = j = k = l$ and $D_{ijkl} = 0$ otherwise. γ is a constant characterising the coupling of the channels which we take equal to 1. Regarding x as a time variable, this problem is a particular case of (1) with $H_{i,i-1} = H_{i,i+1} = 1$, $H_{i,i} = E - 2$ where E is the energy and all the other H_{ij} are equal to zero. An obvious generalisation of these equations in three dimensions is obtained by indexing Ψ by two indices and by discretising the space in the two directions perpendicular to x. Let us recall that the energy levels of H_0 are

$$\varepsilon_k = E - 4\gamma \sin^2(2\pi k/N) \qquad k = 1, \ldots, N \qquad \text{in two dimensions}$$

$$\varepsilon_{k,p} = E - 4\gamma[\sin^2(2\pi k/N) + \sin^2(2\pi p/N)] \qquad k, p = 1, \ldots, N \qquad \text{in three dimensions.}$$

One way of calculating the Kolmogorov entropy Σ is to use the multichannel generalisation of the Thouless theorem [13] asserting that $\Sigma(E) + \mathrm{i}N(E)$, where $N(E)$ is the integrated density of states, is an analytical function of E. As a consequence, one can deduce $\Sigma(E)$ from $N(E)$. On the other hand, one can compute $N(E)$ perturbatively from the Green function of the problem. In the following we present an alternative but more direct method for calculating the Kolmogorov entropy.

The equations of motion are

$$\frac{\partial H}{\partial Q_i} = -\dot{P}_i \qquad \frac{\partial H}{\partial P_i} = \dot{Q}_i .$$

Equivalently, we introduce the evolution operator U of the system, which satisfies

$$\dot{U} = \begin{pmatrix} 0 & I \\ -H_0 + V & 0 \end{pmatrix} U. \tag{4}$$

It is easy to see that one can find a basis where the equation of motion of U is

$$\dot{U} = \begin{pmatrix} -\mathrm{i}\Omega & 0 \\ 0 & \mathrm{i}\Omega \end{pmatrix} U + \frac{\mathrm{i}}{2} \begin{pmatrix} W & W \\ -W & -W \end{pmatrix} U \tag{5}$$

where we have set $\Omega = H_0^{1/2}$ and $W = \Omega^{-1/2} V \Omega^{1/2}$.

This basis can also be chosen so that H_0 is diagonal. The first term of (5) may be eliminated by working in a rotating basis (with 'pulsation' Ω). Writing in this basis the evolution operator

$$U = \begin{bmatrix} X^- & Y^- \\ X^+ & Y^+ \end{bmatrix}$$

where X^ε, Y^ε ($\varepsilon = +1$ or -1) are $N \times N$ matrices, we obtain

$$\begin{aligned} \dot{X}^- &= \tfrac{1}{2}\mathrm{i} W^{-+} X^- + \tfrac{1}{2}\mathrm{i} W^{--} X^+ \\ \dot{X}^+ &= -\tfrac{1}{2}\mathrm{i} W^{+-} X^+ - \tfrac{1}{2}\mathrm{i} W^{++} X^- \end{aligned} \tag{6}$$

where $W^{\varepsilon\varepsilon'} = \exp(\mathrm{i}\varepsilon\Omega t) W \exp(\mathrm{i}\varepsilon'\Omega t)$.

In order to compute the Kolmogorov entropy we argue that

$$\Sigma = \tfrac{1}{2}\langle \partial_t \log\det(X^+ X^-)\rangle. \tag{7}$$

Using the Wronski identity and equations (6) we obtain

$$\Sigma = -\mathrm{Re}\langle \mathrm{i}\,\mathrm{Tr}\, W^{++} A\rangle \tag{8}$$

with $A = X^-(X^+)^{-1}$. It is straightforward to see that A satisfies

$$\dot{A} = \tfrac{1}{2}\mathrm{i} W^{-+} A + \tfrac{1}{2}\mathrm{i} W^{--} + \tfrac{1}{2}\mathrm{i} A W^{+-} + \tfrac{1}{2}\mathrm{i} A W^{++} A. \tag{9}$$

Now owing to the fact that V is a white noise we have for any continuous function $F(A)$ of A

$$\left\langle V_{ij}(t) \int_0^t V_{kl}(t') F(A)\,\mathrm{d}t' \right\rangle = \tfrac{1}{2} D_{ijkl} \langle F(A)\rangle. \tag{10}$$

Using this property one can compute Σ pertubatively in D. The term of order D is easy to obtain. Indeed we have

$$\langle \mathrm{Tr}(W^{++}A)\rangle = \left\langle \mathrm{Tr}\, W^{++}\left(A(0) + \frac{\mathrm{i}}{2}\int_0^t (W^{-+}A + W^{--} + A W^{+-} + A W^{++} A)\,\mathrm{d}t\right)\right\rangle. \tag{11}$$

Therefore

$$\langle \mathrm{Tr}(W^{++}A)\rangle = \frac{\mathrm{i}}{4}\lim_{T\to\infty}\frac{1}{T}\int_0^T (\overline{W^{++}W}^{-+}A + \overline{W^{++}W}^{--} + \overline{W^{++}AW}^{+-} + \overline{W^{++}AW}^{++}A)\,\mathrm{d}t. \tag{12}$$

To the lowest order, only one term contributes (i.e. the term with no A). Hence at order D one finds

$$\Sigma = \tfrac{1}{8}\,\mathrm{Tr}(\overline{WW}). \tag{13}$$

One can go beyond this lowest-order result by a systematic and recursive method. As an example, we sketch here the computation of order D^2. At that order the three other terms of (10) contribute. We proceed in three steps. First, we integrate by parts. For example the term $I_1 = \langle \overline{W^{++}W}^{-+}A\rangle$ gives

$$I_1 = -\left\langle \dot{A}\int_0^t \overline{W^{++}W}^{-+}\,\mathrm{d}t\right\rangle.$$

The second step is then to replace A by its expression (7). For I_1 this gives

$$I_1 = -\frac{\mathrm{i}}{2}\left\langle (W^{-+}A + W^{--} + AW^{+-} + AW^{++}A)\int_0^t \overline{W^{++}W}^{-+}\,\mathrm{d}t\right\rangle. \tag{14}$$

In the third step, we replace A by formally integrating (7). This allows us to use (8) to contract over the noise and separate all the contributions of order D^2.

Proceeding in this way for all the terms of order D^2 which contribute to $\mathrm{i}\langle\mathrm{Tr}(W^{++}A)\rangle$, one sees that all these contributions are purely imaginary. Therefore there is no contribution at order D^2 to the Kolmogorov entropy (note that this is a very general result).

One can proceed further and construct the expansion of Σ in powers of D recursively.

At order n one has to perform the three steps outlined above, so as to obtain the order $n+1$. Note that this algorithm leads to two kinds of terms according to the signs in the exponential it contains: (i) terms with the same number of positive ε and negative ε, which *do not contain* A anymore and (ii) terms with a different number of positive ε and negative ε which contain A but which contribute only at a higher order than terms of the first kind. It is for that reason that our expansion is indeed possible.

We now wish to investigate the thermodynamical limit of the largest Lyapunov exponent $\lambda_{\max}(N)$. The calculation of $\lambda_{\max}(N)$ for an arbitrary number N of channels seems to be a very difficult task. As we will show here, it is nevertheless possible to obtain this exponent for a weak noise in the large N limit by extracting the dominant contribution in the $1/N$ expansion of $\lambda_{\max}(N)$. Note also, that for two channels, a complete derivation is possible in the weak noise limit [14, 15].

In the following $\mu(P, Q, t)$ will denote the measure associated with the system. μ satisfies the Fokker-Planck equation

$$\partial_t\mu + P_i\partial_{Q_i}\mu - H_{0ij}Q_j\partial_{P_i}\mu - \tfrac{1}{2}D_{ijkl}\partial_{P_i}Q_jQ_l\partial_{P_k}\mu = 0. \tag{15}$$

We denote by S the quadratic form

$$S = A_{nm}Q_nQ_m + \tfrac{1}{2}B^{+}_{nm}(Q_nP_m + Q_mP_n) + \tfrac{1}{2}B^{-}_{nm}(Q_nP_m - Q_mP_n) + C_{nm}P_mP_n. \tag{16}$$

A, B^+ and C are symmetric $N\times N$ matrices while B^- is an $N\times N$ antisymmetric matrix. These matrices will be specified later.

The largest Lyapunov exponent is

$$\lambda_{\max}=\tfrac{1}{2}\partial_t\langle\log S\rangle \tag{17}$$

where the mean is taken over the measure. Using the Fokker-Planck equation one obtains

$$\partial_t\langle\log S\rangle=\langle(1/S)P_i\partial_{Q_i}S\rangle-H_{ij}\langle(1/S)Q_j\partial_{P_i}S\rangle+\tfrac{1}{2}D_{ijkl}\langle Q_jQ_l\partial_{P_k}(1/S)\partial_{P_i}S\rangle. \tag{18}$$

After a straightforward calculation one sees that $\partial_t\langle\log S\rangle$ may formally be written as

$$\partial_t\langle\log S\rangle=\langle M(P,Q)/S\rangle+\langle K(P,Q)/S^2\rangle \tag{19}$$

where M is a quadratic form of its variables and K is a quartic expression of Q and P.

Thanks to the homogeneity of the terms in the right-hand side, in the limit of $t\to\infty$, the mean values can be taken with respect to the invariant measure. This invariant measure is the projection of μ on the hypersphere in the space (Q, P).

Now it is possible to choose A, B^+, B^- and C such that there exists ν satisfying

$$M(P,Q)=\nu S. \tag{20}$$

One sufficient condition is that these matrices obey the following set of equations:

$$\begin{aligned}\nu C_{nm}&=B^+_{nm}\\ \nu B^+_{nm}&=2A_{nm}-(H_{mj}C_{nj}+H_{nj}C_{mj})\\ \nu B^-_{nm}&=H_{nj}C_{mj}-H_{mj}C_{nj}\\ \nu A_{nm}&=-\tfrac{1}{2}(H_{nj}B^+_{jm}+H_{mj}B^+_{jn}-H_{jm}B^-_{jn}-H_{jn}B^-_{jm})+D_{nmkl}C_{kl}.\end{aligned} \tag{21}$$

The eigenvalue ν is determined by

$$\begin{aligned}\nu^2C_{nm}=(2/\nu)[&-\tfrac{1}{2}\nu H_{mj}C_{nj}-\tfrac{1}{2}\nu H_{nj}C_{mj}+\tfrac{1}{2}(1/\nu)H_{mj}(H_{nk}C_{jk}-H_{jk}C_{nk})\\ &+\tfrac{1}{2}(1/\nu)H_{nj}(H_{mk}C_{jk}-H_{jk}C_{mk})+\tfrac{1}{2}D_{ijkl}C_{kj}+\tfrac{1}{2}D_{njmk}C_{kj}]\\ &-H_{mj}C_{jn}-H_{nj}C_{mj}.\end{aligned} \tag{22}$$

The key points for the determination of ν in the limit $N\to\infty$ are as follows.

(i) One verifies that the eigenproblem which determines S is the same as the one which gives the asymptotic evolution of the different second-order moments. This evolution is characterised by 'the generalised Lyapunov exponent of order two' defined as $L(2)=\tfrac{1}{2}\lim\log\langle\|\Phi\otimes\Phi\|\rangle$.

(ii) With this choice of S the second term in (18) is negligible. Indeed, at least for the Anderson problem (and even for a wider class of noise structure that we will not investigate in this letter), this term vanishes in the thermodynamical limit.

Let us demonstrate explicitly this last property in the case of the Anderson localisation. The resolution of the eigenproblem leads to the implicit equation

$$L(2)=\tfrac{1}{2}\nu=\tfrac{1}{4}DR_{00}(\nu^2/4) \tag{23}$$

where R is the resolvant $R(z)=1/(H+z)$.

We write (18), $\lambda_{\max}=L(2)(1-\delta)$. For weak noise the sole contribution to δ is

$$\delta=\frac{1}{R_{00}}\sum_{i,j}\left\langle\frac{Q_i^2(C_{ij}P_j)^2}{S^2}\right\rangle \tag{24}$$

with $S = Q_i Q_j + P_i C_{ij} P_j$. After the change of variable $Y = CP$, (24) may be written as

$$\delta = \frac{1}{R_{00}} \sum_i \left\langle \frac{Q_i^2 Y_i^2}{(Q_j Q_j + Y_j H_{jk} Y_k)^2} \right\rangle. \tag{25}$$

In order to prove that δ is negligible in the limit $N \to \infty$, we make the following remarks.

(i) The maximal value of δ is 1. This corresponds to invariant measures sharply peaked along the directions (Q_i, P_i). This situation is indeed realised for N channels totally decoupled. This can be checked by considering the product of N individual log-normal measure (see for instance [16]).

(ii) When the channels are coupled, and at weak noise, the peaks of the invariant measure no longer exist. Indeed, thanks to the ergodicity, mean values can be evaluated by integrating over the time evolution. Even if we start from the worse initial conditions, namely all the (Q_i, P_i) except one equal to zero, the time evolution which can be considered as free on timescales less than $1/\nu$, will isotropise the invariant measure provided ν is less than the spectral width Δ of H_0. This is an effect of *phase mixing*. Notice that the condition $\nu < \Delta$ is indeed sufficient. It is important for the validity of our proof that the much stronger condition $\nu \ll \Delta E$, where $\Delta E \sim \Delta / N$ is the mean level spacing, is not required. A rough estimate of the free evolution of δ shows that *in the thermodynamical limit* it decreases as $1/t$. Hence, one concludes that

$$\lambda_{\max}(N) = L(2) \tag{26}$$

up to a correction of order $L(2)/\Delta$.

Of course, a more direct verification of our claims should be done by computing $L(4)$ and showing explicitly (in the thermodynamical limit) that $L(4) = L(2)$. Then the convexity and the increasing of $L(N)$ would imply $L(2) = \lambda_{\max}$. Unfortunately this calculation seems rather involved.

All the previous results can be applied to the Anderson localisation. The Kolmogorov entropy and ν up to order D^2 are computed to be

$$\Sigma = \frac{D}{8N} \left(\sum_k \frac{1}{\sqrt{\varepsilon_k}} \right)^2 \tag{27}$$

and

$$\lambda_{\max} = \frac{D}{4N} \sum_k \frac{1}{\varepsilon_k}. \tag{28}$$

For $N = 1$, (27) gives the usual result of the one-dimensional case at weak disorder.

A useful quantity for studying the distribution of the Lyapunov exponents is the 'convexity index of the distribution' that we define as: $I = \lim_{N \to \infty} N\lambda_{\max}/\Sigma$. For the triangular distribution $I = 2$. In the general case, I gives some insight into the convexity of the spectrum. At weak disorder, (27) and (28) gives (for $E > 4$ in two dimensions or $\varepsilon > 8$ in three dimensions):

$$I = 2 \int \frac{\rho(\varepsilon)}{\varepsilon} \mathrm{d}E \int \rho(\varepsilon) \, \mathrm{d}\varepsilon \left(\int \frac{\rho(\varepsilon)}{\sqrt{\varepsilon}} \mathrm{d}\varepsilon \right)^{-2} \tag{29}$$

where $\rho(\varepsilon)$ is the density of states associated with H_0. Equation (29) is the central result of this letter. It follows clearly from this formula (by the Schwarz inequality) that for weak disorder I is always greater than two. This result supports the idea that the Lyapunov spectrum is always convex at weak disorder. This property was already noted [9, 10] through numerical simulations. Note that although (29) is proved here under cyclic boundary conditions it must be true independently of this particular geometry.

At high energy in two and three dimensions we obtain $I=2$ from (27) and (28). This result is in agreement with the result of Dorokhov [12].

At lower energy, I depends on E and D and generally $I=2$. Let us look for instance in three dimensions for $E=8$. One has

$$\Sigma=\alpha ND/8 \tag{30}$$

where α is a number independent of D

$$\alpha=\left(\int_0^{2\pi}\frac{\mathrm{d}k_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}k_2}{2\pi}\frac{1}{(\cos^2 k_1+\cos^2 k_2)^{1/2}}\right)^2.$$

On the other hand, ν is given by

$$\nu=\frac{D}{2}\int_0^{2\pi}\frac{\mathrm{d}k_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}k_2}{2\pi}\frac{1}{\nu^2+\cos^2 k_1+\cos^2 k_2}. \tag{31}$$

The ν^2 in the denominator of the integrand makes the result convergent and gives for ν a behaviour: $\nu \sim ND\beta \ln (D_0/D)$ where D_0 and β are some numbers which depend on E. One therefore concludes that I depends logarithmically on D. To make the result more quantitative in two dimensions, one has to take care of the infrared logarithmic divergence of the integral in (27). This question is beyond the scope of this letter.

In this work we have been able to investigate a simple characteristic of the Lyapunov spectrum, namely its 'convexity index', in the weak noise limit. According to our result the triangular law is not recovered, except at weak coupling, in the thermodynamical limit. An important result we have obtained is that in this limit, the largest Lyapunov exponent and the generalised Lyapunov exponent of order two are equal. This absence of dispersion is in notable contrast with the one-dimensional case, where $\lambda_{\max}=\frac{1}{2}L(2)$. We remark that it is precisely this absence of dispersion which yields $I=2$ in the high-energy limit.

Physical quantities, like conductance of samples, are related to the whole Lyapunov spectrum and, particularly for long samples, to the density of the lower exponents. Such information is only available through an infinite number of integral indices characterising the Lyapunov spectrum. The study of such quantities is under investigation using similar techniques.

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